MA 1118 - Multivariable Calculus
Exam I - Quarter I - AY 02-03

Instructions: Work all problems. Read the problems carefully. Show appropriate work, as partial credit will be given. One Page, 8-1/2 by 11, hand-written, one-side notes permitted. No scientific calculators or other notes or tables allowd.

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1. (25 Points) Determine whether the given sequence converges or diverges. If the sequence diverges, state why. If the sequence converges, find the limit.

(a) $$a_n = \frac{n^3+3n+1}{(n^2+1)(4n+2)}$$

solution:

Basically, all we need to do is observe that, for "large" n,

$$a_n = \frac{n^3+3n+1}{(n^2+1)(4n+2)} \to \frac{n^3}{(n^2)(4n)} = \frac{n^3}{4n^3} = \frac{1}{4}$$

i.e. $\lim_{n\to\infty} a_n = \frac{1}{4}$. (We could also determine this using L'Hospital's rule.)

(b) $$a_n = \frac{n}{\ln(1+e^{3n})}$$

solution:

If we observe that, by definition, $x = \ln(e^x)$, and so, for "large" n,

$$a_n = \frac{n}{\ln(1+e^{3n})} \to \frac{n}{\ln(e^{3n})} = \frac{n}{3n} = \frac{1}{3}$$

we would be done, i.e. $\lim_{n\to\infty} a_n = \frac{1}{3}$. However, seeing this may require a bit of luck as well as skill. What should be clear is that, for "large" n, $a_n \to \infty/\infty$, and therefore we can apply L'Hospital's rule directly. Specifically

$$\begin{aligned}\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{x}{\ln(1+e^{3x})} &= \lim_{x\to\infty} \frac{1}{\frac{3e^{3x}}{1+e^{3x}}} \\ = \lim_{x\to\infty} \frac{1+e^{3x}}{3e^{3x}} &= \lim_{x\to\infty} \frac{3e^{3x}}{9e^{3x}} = \frac{3}{9} = \frac{1}{3}\end{aligned}$$

which, of course, is exactly the same result.

(c) $a_n = n^{\ln(n)}$

solution:

In this case, for “large” n,

$$a_n = n^{\ln(n)} \ \rightarrow \ \infty^{\infty} = \infty$$

(Note this is **not** indeterminate!) Therefore $\lim_{n\to\infty} n^{\ln(n)} = \infty$, i.e. the limit does not exist!

2. (25 Points) Determine whether each of the following series converges absolutely, converges conditionally, or diverges. (Identify the test(s) used in each case):

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+4}}$

solution:

Observe that this is an alternating series, and that

$$|a_n| = \frac{1}{\sqrt{n^2+4}} \to \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test. However, clearly

$$|a_{n+1}| = \frac{1}{\sqrt{(n+1)^2+4}} < |a_n| = \frac{1}{\sqrt{n^2+4}} \quad \text{and} \quad \lim_{n\to\infty} |a_n| = 0$$

and therefore, by the alternating series test, this series converges (but only conditionally, in view of our earlier result).

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+3}$

solution:

Observe this is a series of nonnegative terms. Moreover, for "large" n,

$$a_n = \frac{\sqrt{n}}{n^2+3} \to \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

But $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by the p-test, since $p = 3/2 > 1$. Therefore the original series converges (absolutely, since the terms are already nonnegative) by the limit comparison test.

(c) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \, \frac{k^2+1}{k^2+4}$

solution:

Observe that, for "large" k,

$$a_k = (-1)^k \, \frac{k^2+1}{k^2+4} \quad \to \quad (-1)^k \, \frac{k^2}{k^2} = (-1)^k \neq 0$$

Therefore, the series must diverge.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

solution:

Observe

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \qquad \Longrightarrow \qquad a_n = \frac{(n!)^2}{(2n)!} > 0$$

In this case, its not obvious how a_n 'looks" for "large" n. Therefore, because of the factorials, we should try the ratio test. Proceeding

$$\frac{a_{n+1}}{a_n} = \frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \frac{(n+1)! \; (n+1)! \; (2n)!}{n! \; n! \; (2n+2)!} = \frac{(n+1)(n+1)}{(2n+2)(2n+1)}$$

Therefore

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

and so by the ratio test, the series converges (and absolutely, since the terms are already positive).

3. (10 Points) Find the radius and *open* interval of convergence of the Series

$$\sum_{n=2}^{\infty} \frac{e^n}{n^2} (x+1)^n$$

solution:

This has the form of a standard power series, i.e.

$$\sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{with} \quad c_0 = c_1 = 0 \quad , \quad c_n = \frac{e^n}{n^2} \ , \ n \geq 2 \quad , \quad a = -1$$

Therefore, the radius of convergence is

$$\rho = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \to \infty} \left| \frac{\frac{e^n}{n^2}}{\frac{e^{n+1}}{(n+1)^2}} \right| = \lim_{n \to \infty} \frac{(n+1)^2}{n^2 \ e}$$
$$= \frac{1}{e} \lim_{n \to \infty} \frac{(n+1)^2}{n^2} = \frac{1}{e} \ (1) = \frac{1}{e}$$

Therefore, $\rho = 1/e$ and the series converges for

$$|x+1| < \frac{1}{e} \qquad \Longrightarrow \qquad -1 - \frac{1}{e} < x < -1 + \frac{1}{e}$$

4. (25 Points) a. Write the first three terms (i.e. up through $n = 2$) of the MacLaurin series expansion for:

$$f(x) = \ln(1 + 3x).$$

solution:

For a MacLaurin series, we have, by definition, $a = 0$, and so we can then create the table

n	$f^{(n)}(x)$	$f^{(n)}(a)$	c_n
0	$\ln(1+3x)$	$\ln(1) = 0$	0
1	$\frac{3}{1+3x}$	$\frac{3}{1+3(0)} = 3$	3
2	$-\frac{9}{(1+3x)^2}$	$-\frac{9}{(1+3(0))^2} = -9$	$-\frac{9}{2!} = -\frac{9}{2}$
3	$\frac{54}{(1+3x)^3}$	$\frac{54}{(1+3(0))^3} = 54$	$\frac{54}{3!} = 9$

Therefore the first three terms of the MacLaurin series are

$$f(x) = \ln(1 + 3x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots = 3x - \frac{9}{2}x^2 + \cdots$$

b. What would be the maximum error you would expect to find if you were to use your answer to part a to estimate $\ln(1.15)$.

solution:

According to the Taylor Remainder theorem, if we approximate

$$f(x) = \ln(1 + 3x) \doteq 3x - \frac{9}{2}x^2 \qquad \Longrightarrow \qquad R_2(x) = \frac{f^{(3)}(\xi)}{3!}x^3$$

where $0 < \xi < x$. But note, from the above table

$$f^{(3)}(\xi) = \frac{54}{(1 + 3\xi)^3} \qquad \Longrightarrow \qquad 0 < f^{(3)}(\xi) \le 54 \text{ for } 0 \le \xi \le x$$

Moreover, for $f(x) = \ln(1 + 3x)$,

$$f(x) = \ln(1.15) \qquad \Longrightarrow \qquad (1 + 3x) = 1.15 \qquad \Longrightarrow \qquad x = 0.05$$

solution:

Therefore, by the above discussion

$$R_2(0.05) = \frac{f^{(3)}(\xi)}{3!}(0.05)^3 \leq \frac{54}{3!}(0.05)^3 = 9\,(0.05)^3 = 0.001125$$

Note

$$\ln(1.15) = .13976194\ldots \quad \text{while} \quad 3(0.05) - \frac{9}{2}(0.05)^2 = .13875000$$

and therefore the actual error is .00101194..., which is lower than the maximum computed above.

5. (15 Points) Find the sum of each of the following series:

(a) $\displaystyle\sum_{n=0}^{\infty} 3^{-n} x^{2n}$

solution:

Observe there are no factorials in the denominator here, so there is little chance this is a "relative" of either the exponential, sine or cosine. By elimination, that leaves only the geometric. With this as a goal, we can fairly easily see that

$$\sum_{n=0}^{\infty} 3^{-n} x^{2n} = \sum_{n=0}^{\infty} \frac{\left(x^2\right)^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{x^2}{3}\right)^n$$

which is precisely the geometric series, with r replaced by $x^2/3$, i.e.

$$\sum_{n=0}^{\infty} 3^{-n} x^{2n} = \frac{1}{1 - \frac{x^2}{3}} = \frac{3}{3 - x^2}$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{3n}}{n!}$

solution:

Here the factorial in the denominator strongly suggests that this is a "relative" of the exponential, although we'll also have to address the fact that this series does not start with $n = 0$. With these observations, we can fairly easily see that

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{3n}}{n!} = \sum_{n=1}^{\infty} \frac{\left(-x^3\right)^n}{n!}$$

which is, except for the starting index, precisely the Taylor/MacLaurin series for e^{x^3}. That minor problem, however, is easily fixed, i.e.

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{3n}}{n!} = \sum_{n=1}^{\infty} \frac{\left(-x^3\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\left(-x^3\right)^n}{n!} - 1 = e^{-x^3} - 1$$